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Finite size effect on the current enhancement at the tip of a long defect in a resistor network

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Abstract. The current at the tip of a long defect in a resistor network has been investigated as a function of the defect length and of the network size. Numerical simulations show that noticeable size effects influence the tip current. In particular, the tip current as a function of the defect length is shown to converge very slowly toward the asymptotic square root law previously predicted for large defects in an infinite network. The nature of the corrections to this asymptotic law is exhibited with a finite size scaling analysis. For practical use, an empirical formula for the tip current is proposed.

1. Introduction

The influence of the finite system size on the stress field in the vicinity of a crack tip has been much studied in the mechanical context (Chona *et al* 1983, Sih 1973, Bowie 1973). This is a difficult problem which crucially depends on the precise geometry of the structure and on the chosen boundary conditions. Specific mathematical techniques have been developed, mainly in the mechanical context, for the analysis of the stress field of an internal crack in a finite geometry (see Sih 1973 for a review). An electrical version of this problem is obtained by considering a discrete resistor network in which a thin row of insulating bonds is equivalent to a crack. As its mechanical counterpart, the resistor network is described by a Laplacian field but offers two main simplifications: the field is scalar instead of being tensorial and its discrete nature facilitates the numerical computation of the field.

In the present work, we have considered a particular crack configuration in such a resistor network. It consists in a long thin row of insulating bonds oriented perpendicular to the average current as shown in figure 1. Such a critical defect has

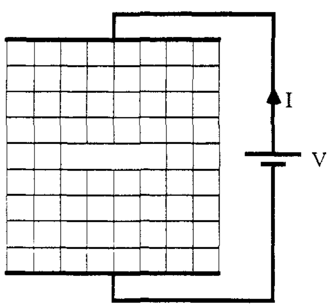

Figure 1. Single defect centred in a resistor network.

been recently examined by Duxbury *et al* (1986, 1987a) and Li and Duxbury (1987) in the context of the theory of breakdown in random networks. In order to estimate the current enhancement at the tips of such a defect, they calculated the solution for the current at the end of an elliptical defect in a continuous 2D conductor and obtained the solution for the discrete model by integrating the current density over the lattice spacing. On doing this, they found that, in two dimensions, the current at the tip of a long defect of length n in a large system grows as†

$$i_{\mathrm{tip}} \sim 1 + f n^{1/2} \tag{1}$$

where f is a numerical factor ($f \sim 0.55$ in a discrete network according to Kahng *et al* 1988).

This expression has been established in the limit of large n in an infinite system (of size $L/n \to +\infty$). Therefore, it is natural to ask about its range of validity since experimental or numerical systems are finite. In the context of the random fuse model, equation (1) has been used to predict the failure threshold in finite systems and for finite crack configurations. Recently, Kahng *et al* (1988) reported a numerical simulation of $i_{\mathrm{tip}}(n)$ on a 91×91 discrete network and noticed that the square-root law was not evident until large values of n ($n > 30$). Due to possible crossover effects, they questioned the utility of calculations based on this law for the range of network sizes ($L \sim 100$) used in numerical simulations.

For the purpose of clarifying the importance of finite size effects in resistor networks, we have investigated the current at the tip of such long defects as a function of their length n and of the network size L. Two finite-size effects are competing in finite networks. For small defect lengths n, the finiteness of n is the dominant effect (effect 1). In this regime, the current i_{tip} converges slowly towards the asymptotic square-root law (1) as the defect length n increases. Analysis of the defect finite size provides corrections going as $n^{-1/2}$. This result stays valid as long as the condition $n \ll L$ is fulfilled. As n increases, the finite size of the network comes into play (effect 2) and rapidly supersedes effect 1. In this case, it can be shown that the leading correction to the asymptotic expression (1) is proportional to $(n/L)^2$ due to the dipole nature of the defect, as seen from the border of the system. In numerical simulations, effect 2 is clearly indicated by the divergence of the two curves $i_{\mathrm{tip},I}(n)$ and $i_{\mathrm{tip},V}(n)$ in current- and voltage-biased networks respectively. The fact that this network size effect is numerically observed for values of n/L as small as 0.10 as discussed later, has important consequences. For small or moderate size networks, say $L < 100$, either effect 1 or effect 2 affects i_{tip} whatever the length of the defect. The apparent square-root law that can be observed in some ranges of defect lengths in such networks cannot really be identified with the asymptotic law predicted in (1). In fact, very large networks are needed to observe the true asymptotic square root law predicted by Duxbury *et al* (1987*b*).

The outline of the paper is as follows. In section 2, numerical simulations of i_{tip} as a function of n and L are presented in order to exhibit the size effects which influence the tip current. In section 3, the corrections to the asymptotic law (1) are predicted

† In order to obtain the square-root dependence, Duxbury *et al* (1986, 1987a), following a suggestion of Stephen, have solved the analogous continuum problem of an elliptic defect of length n and width b. By solving the Laplace equation in this system with the boundary condition $\partial V/\partial \xi = 0$ (ξ is the normal coordinate on the ellipse) leads to a current density at the ellipse tip $j_{\mathrm{tip}} \sim jn$. Taking $b = h$, where h is the lattice mesh size of the corresponding discrete problem and integrating j_{tip} from the ellipse tip to a distance equal to h from this tip yields the square-root singularity. However, to our knowledge, no direct derivation exists for the discrete lattice.

within a finite-size scaling analysis and compared with numerical simulations. An approximate expression for $i_{\rm tip}$ in a finite network is given. The results are summarized in section 4.

2. Numerical results

The system under study is a 2D square resistor network of size L where L is the number of columns and lines (figure 1). Each bond has the same resistance R. A constant current I or a constant voltage difference V is applied between the two busbars located at the top and at the bottom of the network. In the following, the currents (or voltages) are normalized to the current $I_0 = I/L$ (or voltage $V_0 = V/L$) which passes through each vertical bond in the uniform network. The defect we consider is a row of insulating bonds oriented perpendicular to the average current and centred in the network. In order to obtain the current $i_{\rm tip}$ which flows through the critical bonds at the end of the defect, we have performed calculations of the distribution of currents in the network. To solve the Kirchhoff's equations, we used the conjugate-gradient method, the convergence criterion being satisfied when the residual vector was less than 10^{-10} (Duxbury *et al* 1987b).

In the following, both current-biased and voltage-biased networks are considered. In an infinite network, they would give the same behaviour for $i_{\rm tip}$ as a function of the defect length n. In a finite network, the same behaviour is also expected in both cases for small values of n, i.e. as long as the defect does not significantly modify the total conductance of the network. However, as the defect length becomes sufficiently large compared to the size L of the network, the tip current exhibits a stronger enhancement in the current-biased network than in the voltage-biased network. Hence, the curves $i_{{\rm tip},I}(n)$ and $i_{{\rm tip},V}(n)$ are identical for small n but diverge at sufficiently large values of n. The value of n for which the two curves diverge indicates the minimum defect length for which the finiteness of the system length L must be considered (effect 2).

The tip current $i_{\rm tip}$ is displayed in figure 2 as a function of the defect length n for network sizes increasing from $L = 30$ up to $L = 300$. Figures 2(a) and 2(b) correspond

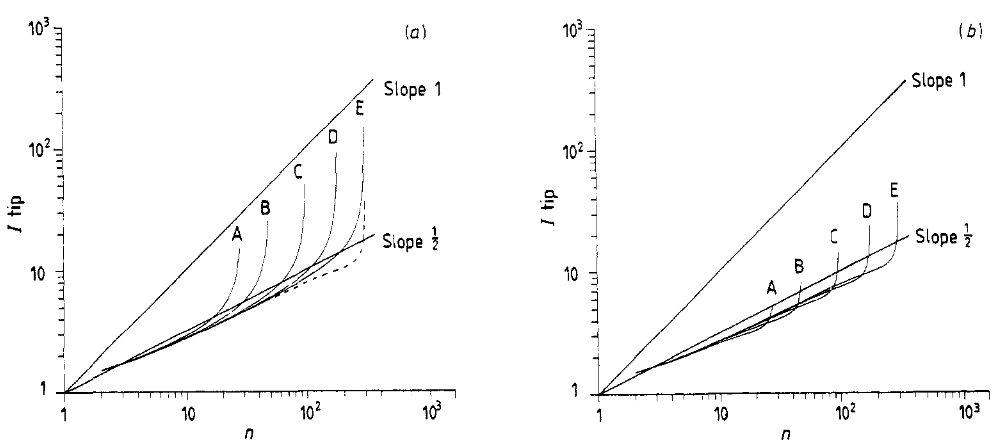

Figure 2. Double logarithmic plot of the tip current $i_{{\rm tip},I}$ in current-biased networks (a) and $i_{{\rm tip},V}$ in voltage-biased networks (b) as a function of the defect length n: (A) $L = 30$, (B) $L = 50$, (C) $L = 100$, (D) $L = 180$, (E) $L = 300$. The tip current $i_{{\rm tip},V}$ in the $L = 300$ voltage-biased network has been displayed in (a) as a broken line for comparison with $i_{{\rm tip},I}$.

to current-biased and voltage-biased networks respectively. The double logarithmic plot that we have used enables one to compare the behaviour of i_{tip} with the straight line corresponding to the square-root law. As discussed above, note the different behaviours of i_{tip} in the current and voltage-biased cases. The voltage-biased curve $i_{\mathrm{tip},V}$ corresponding to the largest network ($L=300$) has been plotted as a broken curve in figure 2(a) for comparison with the corresponding current-biased curve $i_{\mathrm{tip},I}$. The two curves diverge for values of $n \geqslant n^*$ where $n^*(L=300) \sim 40$. Thus, the size of the network affects i_{tip} for rather low values of n ($n^*/L \sim 13\%$). This value, which is found to also hold for the smaller lattices we have investigated, indicates that very large networks are needed to verify the asymptotic form of $i_{\mathrm{tip}}(n)$. For example, in the numerical simulation of $i_{\mathrm{tip}}(n)$ on a 91×91 discrete network, Kahng *et al* (1988) found that the square-root law was not 'evident' until a value larger than $n \sim 30$. With the above criterion, this value $n \sim 30$ is already larger than the crossover value $n^*(L=91) \sim 10$. As a consequence, Kahng *et al* were right to question the validity of (1) in their finite-size network. As discussed later, the apparent numerical confirmation of the square root law for some ranges of defect lengths in small networks can be traced back to a subtle compensation between finite defect and finite system size effects. Noticeable finite-size effects persists even for the largest system we have considered ($L=300$, i.e. $\sim$180 000 resistors). In fact, the logarithmic representation of i_{tip} in figure 2 is not sufficient to obtain conclusive results about the asymptotic behaviour of i_{tip}†.

A more precise analysis of the numerical data can be obtained by displaying (figure 3) the slope $\alpha(n)$ of the logarithmic curves shown in figure 2. This slope corresponds

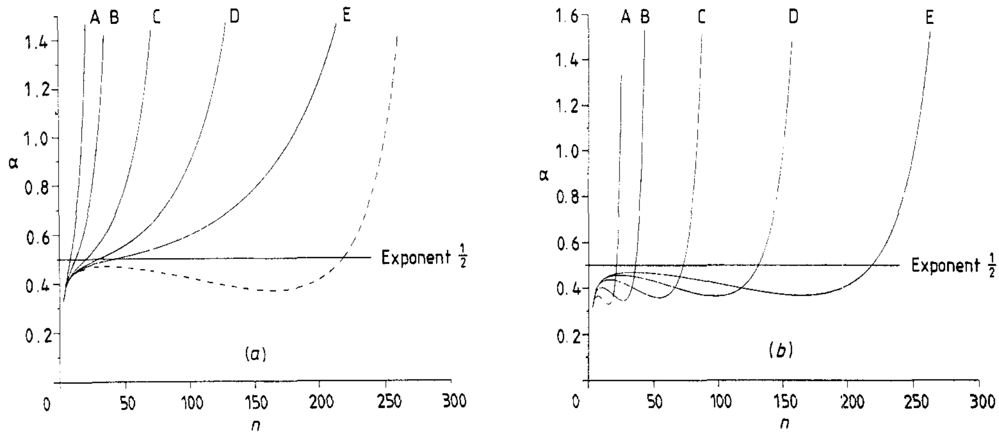

Figure 3. Local effective exponent $\alpha(n)$ for current-biased (a) and voltage-biased networks (b) as a function of the defect length n: (A) $L=30$, (B) $L=50$, (C) $L=100$, (D) $L=180$, (E) $L=300$. The curve corresponding to the $L=300$ voltage-biased network has been displayed in (a) as a broken curve for comparison with the current-biased networks.

† Figure 3 in Kahng *et al* (1988) reports on the same logarithmic representation as we have used, the current enhancement $i_{\mathrm{tip},V}-1$ instead of $i_{\mathrm{tip},V}$ as a function of the defect length for a 91×91 network. Therefore, the displayed curve is somewhat different from our curves, especially for the small values of n. It is worth pointing out that in this logarithmic representation, voltage-biased curves of the current enhancement $i_{\mathrm{tip},V}-1$ seem to better approach the square-root law than the current-biased curves, in contrast with the discussion of our data in figure 2. This fact demonstrates that the logarithmic representation of i_{tip} which seems natural for verifying the square-root law can be misleading. Significantly larger networks than the ones we have considered, would provide unambiguous conclusions about i_{tip} in this representation.

to the local effective power describing $i_{\rm tip}$, i.e. $i_{\rm tip}(n) \propto n^{\alpha(n)}$. Figures 3($a$) and 3($b$) correspond to the current-biased and voltage-biased networks respectively. Three regimes can be identified at small, intermediate and large values of n respectively. For small values of n (regime A), $\alpha(n)$ increases rapidly starting from the initial value $\alpha(3) \sim 0.3$†. In this regime, the behaviour of $i_{\rm tip}$ is dominated by the finite defect length effect (effect 1). For the large values of n (regime C), the behaviour of $i_{\rm tip}$ is dominated by the finite size of the network (effect 2). For current-biased networks figure 3(a), $i_{{\rm tip},I}$ increases rapidly with the defect length n. For the voltage-biased case figure 3(b)), the effect of the finite size of the network is more complicated. In fact, the effective power $\alpha_V(n)$ starts decreasing before the fast increase at the largest values of n. Finally, regime B corresponds to the transition range between regimes A and C. In this range, both effects 1 and 2 are competing but should vanish when $L \to \infty$. This is the range of defect lengths where it is possible to observe the asymptotic square-root law by verifying that $\alpha(n) \to 0.5$ when L increases.

The interest of this representation is to clearly exhibit the interplay between the three regimes. First, a careful comparison between figures 3(a) and 3(b) shows that the current-biased curves and the corresponding voltage-biased curves diverge for the defect length $n^* \sim L/10$. Therefore, effect 2 affects $i_{\rm tip}$ over 90% of the defect length range. Because of the wide range of regime C, regimes A and B spread over a very small defect length range, especially in small networks. In fact, the range of regime B is so small in the $L = 30$ and $L = 50$ networks that it is not possible to distinguish any tendency for $\alpha(n)$ to saturate toward the value 0.5. This tendency timidly shows up in the $L = 100$ network and is more clearly observed in the $L = 180$ and $L = 300$ networks. These results corroborate our previous statement: in the range of defect lengths where a square-root law is observed in small or moderate size networks (say $L < 100$), both effects 1 and 2 are still noticeably competing.

To summarize this section, the above numerical data have illustrated the importance of the size effects. Therefore, in the moderate size networks usually considered in experiments or numerical simulations, it would be useful to know the first finite-size correcting terms when using (1). The next section will be devoted to a more detailed account of these corrections.

3. Finite-size effect corrections

Clearly, the finite-size effects are still noticeable in our networks. Using (1) in a significant and practical range of defect lengths would involve systems of huge sizes ($L > 10^3$), a challenge out of usual computing abilities. Hence, one needs to take the size effects corrections to (1) into account for systems having reasonable sizes. As previously discussed, it is convenient to distinguish:

(i) the case where $n \ll L$ for which the finiteness of n is the dominant effect (effect 1);

(ii) the regime where n is no longer small compared with L (in fact, the criterion found numerically gives $n/L \geqslant 0.10$), for which $i_{\rm tip}$ contains additional correction terms steming from the finite value of n/L (effect 2).

† Only even values of the defect length have been computed in our numerical simulations ($n = 2, 4, \ldots$). Hence, the slopes $\alpha(n)$ displayed in figure 3 are defined for odd values of n according to $\alpha(n) = \log(i_{\rm tip}[2p+2]/i_{\rm tip}[2p])/\log(2)$ where $n = 2p+1$, $p = 1, 2 \ldots$.

3.1. *Case of a finite defect in an infinite system* ($n \ll L$)

In the absence of a direct theoretical treatment for the discrete lattice problem, we will use the formula developed for defects in a continuous system. Then, we apply Duxbury *et al*'s procedure of integrating the continuum result over the lattice spacing (1) in order to obtain formula for the discrete lattice case. The continuous system considered by Duxbury *et al* is an elliptic defect of length a and width b in the limit $a/b \to \infty$ where b is chosen equal to the lattice mesh size. Here, we consider another continuous system, namely a crack in a two-dimensional foil for which the classical results found in the literature (Jackson 1975) allow us to predict not only the leading term for i_{tip} but also size-effect corrections. In particular, the relationship between the square-root law (1) for the current at the tip of a defect in a discrete system and the nature of the singularity in the corresponding continuum system is clearly exhibited.

Consider the electrical problem of a continuous two-dimensional foil with a 'thin crack' of length 1 perpendicular to the electric field at infinity. In the neighbourhood of a crack tip, the electric field and thus the current density can be expressed by a series of terms such as

$$j(r, \phi) = j_0 K_I(l) r^{-1/2} \sum_{k=0}^{+\infty} A_k \sin\{(k+1)\phi/2\}(r/l)^{k/2} \tag{2}$$

where j_0 is the current density very far from the defect. The crack is lying along the positive x axis (the left tip of the crack is at the origin) and (r, ϕ) are the polar coordinates. This expression (2) is a straightforward transcription of the method described in Jackson (1975). Adapted to our boundary conditions, we state that the component of the electric field normal to the crack tips must vanish ($j_\perp(r, \phi = 0$ and $2\pi) = 0$).

The so-called 'current intensity factor' $K_I(l)$ is given by

$$K_I(l) = (\pi l)^{1/2}. \tag{3}$$

This development is valid for small r/l ratio. Sufficiently near the tip, $j(r, \phi)$ is given by the first leading term

$$j(r, \phi) = A_0 j_0 K_I(l) r^{-1/2} \sin\{\phi/2\}. \tag{4}$$

Note the close relationship between the square root dependence of the current intensity factor, the $r^{-1/2}$ singularity† and the square-root law (1) for the current at the tip of a defect in the discrete system.

The $r^{-1/2}$ singularity of the stress field near the crack tip is important in the mechanical context. The stress intensity factor $K_I(l)$ is generally defined by the singular dependence of the stress as one approaches within a distance r from the crack tip. Similar expressions to (2) and (4) exist for the stress field with a 'stress intensity factor' (Ergodan 1983) whose value is exactly given by (3).

Following Duxbury *et al* (1987b) and Li and Duxbury (1987), one can estimate the current in the bond at the tip of the long defect of length l in the corresponding

† The square root singularity $r^{-1/2}$ of the electric (or stress) field at the tip of a crack is a special case of the general problem of a defect having the shape of an edge with opening angle θ. In this case, the $r^{1/2}$ singularity is replaced by a $r^{(1-\pi/\theta)}$ singularity with an exponent $1-\pi/\theta$ which is continuously dependent on the edge angle θ. The 'thin crack' then corresponds to $\theta = 2\pi$ which recovers the exponent $\frac{1}{2}$.

discrete lattice by integrating the current density $j(r, \phi = \pi)$ given by (2) over the lattice bond spacing h, i.e. from $r = 0$ to $r = h$:

$$I_{\text{tip}} = \int_0^h j(r, \phi = \pi)\, \mathrm{d}r \tag{5}$$

$$= I_0\{a_0(l/h)^{1/2} + a_1 + a_2(l/h)^{-1/2} + \ldots\}$$

or

$$i_{\text{tip}} = I_{\text{tip}}/I_0 = a_0 n^{1/2} + a_1 + a_2 n^{-1/2} + \ldots. \tag{6}$$

Here, $n = l/h$ and $a_0, a_1, a_2 \ldots$ are constants determined from the boundary conditions. $I_0 \equiv j_0 h$ is the current per bond at infinity or in the undeteriorated system. We note that all coefficients a_{2m-1} with $m = 1, \ldots + \infty$ are proportional to $\sin\{m\phi\}$ and are thus predicted to vanish exactly for $\phi = \pi$. In particular, $a_1 = 0$. In the following, we will verify this prediction.

Equation (6) recovers the result of Duxbury *et al* (1986, 1987a) and Li and Duxbury (1987), i.e. $i_{\text{tip}} \sim a_0 n^{1/2}$ for large values of n. Furthermore, it provides the form of the leading correction given by $a_2 n^{-1/2}$.

Equation (6) can be compared with the numerical values of i_{tip} discussed in section 2. This comparison is not easy as it can only be made in regime A where effect 2 due to the finite size of the network is negligible. From the discussion in section 2, regime A spreads over an interval of defect lengths $[1; n_1]$ where n_1 is smaller than $L/10$. In practice, it is not possible to separate effects (1) and (2) in the $L = 30$, $L = 50$ networks and one finds a very small value, $n_1 \sim 7$, in the $L = 100$ network. Thus, one needs to consider rather extended networks such as our $L = 180$ and $L = 300$ networks. The comparison was made between (6) truncated to its three leading terms shown in (6), and the numerical values of i_{tip} in the $L = 300$ network. The fit, which could only be obtained for defect lengths $n \leqslant 20$, provides $a_0 = 0.81 \pm 0.01$, $a_1 = 0.00 \pm 0.05$ and $a_2 = 0.51 \pm 0.02$. The smallness of the defect length interval stresses again the necessity to work with large systems in order to clearly separate effects 1 and 2. The value $a_1 = 0.00 \pm 0.05$ provides a first valuable test of our fit since we expect a_1 to vanish exactly. Comparison between the numerical curves of i_{tip} for the $L = 300$ network (figure 2) and (6) where the above values a_0, a_1 and a_2 have been used, is displayed

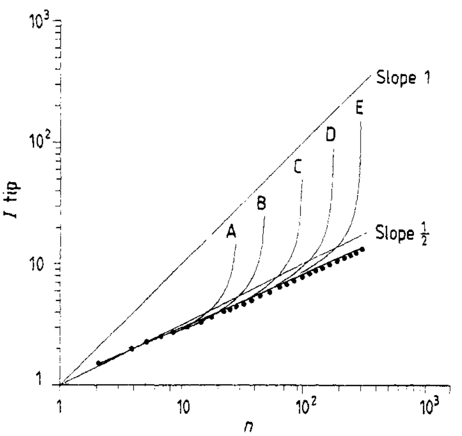

Figure 4. Comparison between the numerical values of i_{tip} (full curves) and (6) (dots). (A) $L = 30$, (B) $L = 50$, (C) $L = 100$, (D) $L = 180$, (E) $L = 300$.

in figure 4. In this logarithmic representation, the agreement is good up to the values of n where the current-biased and voltage-biased curves diverge ($n \sim 40$). The agreement can be checked to be also good up to $n = 20$ when using the more sensitive effective power representation of figure 3.

The above values of a_0, a_1 and a_2 can also be checked from the numerical values of i_{tip} in an infinite lattice computed by using the lattice Green function for monopole sources placed along the defect bonds in the unaltered network (Duxbury *et al* 1987b). Performing a least-squares fit over the interval $n = [1; 225]$ between these values of i_{tip} and the two-term expansion $i_{\mathrm{tip}} = a_0 n^{1/2} + a_2 n^{-1/2}$ provides $a_0 = 0.8076$ and $a_2 = 0.4965$, in good agreement with the values we found in the $L = 300$ network†.

To summarize the results of this section, a reasonable approximation of i_{tip} for $n \leqslant L/10$ in a network of size L is given by:

$$i_{\mathrm{tip}} \sim a_0 n^{1/2} + a_2 n^{-1/2} \tag{7}$$

where $a_0 \sim 0.8$ and $a_2 \sim 0.5$.

The range of validity $n \leqslant L/10$ imposes stringent conditions on the size of a finite network in order to observe the leading scaling behaviour $a_0 n^{1/2}$. Let us assume that the first correction $a_2 n^{-1/2}$ can be neglected. For instance, let us choose $a_2 n^{-1/2} / i_{\mathrm{tip}} \leqslant 5\%$ which is a very loose condition. One finds $n \geqslant 12$ and thereby, $L \geqslant 120$. For $a_2 n^{-1/2} / i_{\mathrm{tip}} \leqslant 1\%$, one would find $L \geqslant 600$.

3.2. *Case of a finite defect in a finite system*

As stated before, the influence of the finite system size on the stress field in the vicinity of a crack tip has been much studied in the mechanical context (Chona *et al* 1983, Sih 1973, Bowie 1973). For instance, Bowie (1973), using the so-called 'modified mapping-collocation' technique, has been able to give the dependence of the stress intensity factor $K_I(l)$ as a function of l/L for a crack in an arbitrary rectangular geometry. For the particular case of a square geometry and fixed external stress (corresponding to the current-biased resistor network), he finds

l/L	0.040	0.100	0.200	0.400	0.500	0.667	0.741	0.800
$K_I(l)/(\pi l)^{1/2}$	1.00	1.01	1.06	1.22	1.33	1.60	1.78	2.00

Note that for values of the ratio l/L as small as 0.1–0.2, there are small but already noticeable corrections to the stress enhancement factor. This l/L ratio is indeed in good agreement with the numerical estimation ($n/L \sim 0.10$) found in our discrete resistor networks.

More specifically, it can be shown (Sih 1973) that the effect of a finite system size L is to introduce corrections to the stress intensity factor, which are function of the sole ratio l/L (Bowie 1973, Bui 1977). In particular, the leading correction is proportional to $(l/L)^2$ in the limit of small l/L‡, a result which has been illustrated for a

† The formula $i_{\mathrm{tip}} = a_0 n^{1/2} + a_2 n^{-1/2}$ where $a_0 = 0.8076$ and $a_2 = 0.4965$ is accurate to a few per cent for $n \leqslant 10$, less than one per cent for $10 < n < 50$ and less than a part in a thousand for $n > 50$. Better fits can be obtained if higher-order terms are considered in (6). In such cases, the values of a_0 and a_2 are slightly modified ($a_0 \sim 0.8066$, $a_2 \sim 0.5876$, $a_4 \sim -0.200, \ldots$). However, the accuracy gained by such higher-order expansions is rapidly superseded by the corrections due to effect 2 in finite lattices.

‡ For instance, let us quote the expression (Ergodan 1983, Bowie 1973) of the stress intensity factor for a crack of length n in a plate of width L and length $H > 3L$, submitted to a constant tensile stress at the two ends of the plate: $K_I(n) = (\pi n)^{1/2}[\cos\{\pi n/4L\}]^{-1/2}\{1 - 0.006(n/L)^2 + 0.015(n/L)^4\}$ which, for small n/L, gives the form of the leading correction to expression (4): $K_I(n) = (\pi n)^{1/2}\{1 + [(\pi^2/64) - 0.006](n/L)^2 + \ldots\}$.

particular goemetry by Bui (1977). Such a $(l/L)^2$ correction comes from the dipole r^{-2} dependence of the stress (or current) field at distances r from the crack much larger than its size l. This result can be checked for the tip current in the discrete resistor network, by separating the (n/L) corrections from the finite defect length corrections given by (6). In particular, it is possible to show that the numerical values of $i_{\text{tip},I}/(a_0 n^{1/2}+a_2 n^{-1/2})$ are reasonably well approximated by the expression $\{1-(n/L)^2\}^{-1/2}$ for the current-biased networks. This result corroborates the predicted $(n/L)^2$ form of the leading correction in the limit $n/L \to 0$ and leads to the following expression for $i_{\text{tip},I}$:

$$i_{\text{tip},I} \sim (a_0 n^{1/2}+a_2 n^{-1/2})/(1-(n/L)^2)^{+1/2}. \tag{8}$$

We emphasize that (8) is only a rough approximation of the numerical values of $i_{\text{tip},I}$. However, the first advantage of this expression is the absence of adjustable parameters in the $(n/L)^2$ correcting term. Moreover, the approximation is reasonably good as demonstrated in figure 5 where (8) is compared with the numerical values of $i_{\text{tip}I}$. Therefore, we suggest using (8) in the practical cases where i_{tip} does not need to be known with a very large accuracy (<1%).

A similar result can be obtained for voltage-biased networks by using

$$i_{\text{tip},V}/i_{\text{tip},I} = \sigma(n/L) \tag{9}$$

where $\sigma(n/L)$ is the conductance of the network. In a continuous system, the conductance is actually function of the sole ratio $1/L$. In a discrete resistor network, there exists a slight additional dependence on the system size L which can be neglected in a first approximation. It is possible to show that the numerical values of $\sigma(N/L)$ are reasonably fitted by

$$\sigma(n/L) \sim 1-b(n/L)^2 \tag{10}$$

valid for $n/L \leqslant 0.9$ and where $b \sim 0.73$. For small values of n/L, this expression could be put on a firm basis in terms of dipole effects. However, as one wants to use it in a rather extended range of defect lengths, we emphasize that equation (10) is only the result of an approximate fit of the numerical values of $\sigma(n/L)$. The predicted values for $i_{\text{tip},V}$ given by (8), (9) and (10) are compared with the actual numerical values in

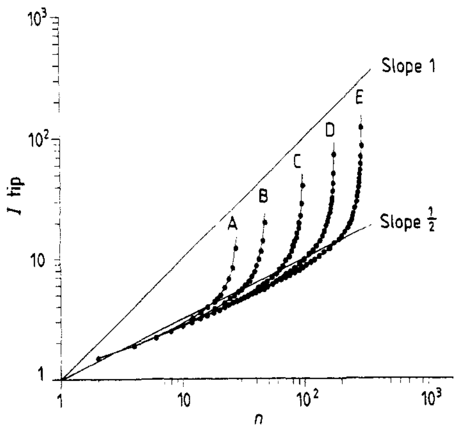

Figure 5. Comparison between the numerical values of $i_{\text{tip},I}$ (full curves) and (8) (dots). (A) $L=30$, (B) $L=50$, (C) $L=100$, (D) $L=180$ and (E) $L=300$.

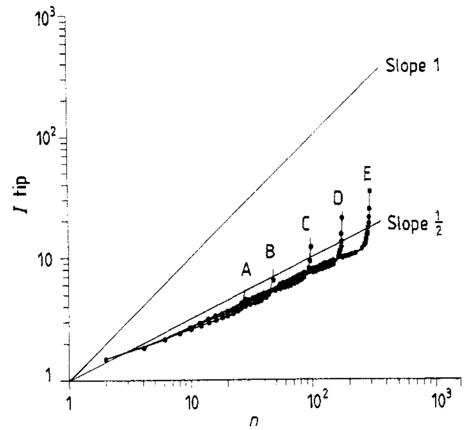

Figure 6. Comparison between the numerical values of $i_{\mathrm{tip},V}$ (full curves) and the approximate values given by (8), (9) and (10) (dots). (A) $L=30$, (B) $L=50$, (C) $l=100$, (D) $L=180$ and (E) $L=300$.

figure 6. Given the new errors introduced by (10), the fit is not as good as the fit of $i_{\mathrm{tip},I}$ but can be sufficient in many practical cases.

In fact, the above results are quite general and concern any system described by the same equations (Laplacian fields). From the previous study, we have shown that in order to reduce the error on i_{tip} to a value less than say, 2%, one needs to consider the defect length range $n/L<10$. Above this value, corrections due the finite size of the system must be considered. This result is closely related to that obtained by Chona *et al* (1983) on the spatial extension r_{min} of the singularity-dominated zone around a crack charged according to different schemes. The value of r_{min} was obtained from systematic series expansion up to the sixth order of the stress field in conjunction with photo-elastic fracture patterns. Depending upon the nature of the boundary stresses, it was shown that r_{min}/L must be less than 0.2–0.1% in order that the stress calculated from the leading singularity be equal to the true stress to within less than a 2% error. The concept of a 'singularity-dominated zone' around a crack tip which is widespread in the literature must thus be taken with care in real specimens with finite dimensions. Our results, obtained in an electrical system, provide another illustration of these effects.

4. Conclusion

We have presented numerical results showing that noticeable finite-size corrections apply to the leading singularity occuring at the crack tip in a two-dimensional electrical problem. In particular, one must be very cautious when a square-root law governing the tip current is observed. For network sizes usually considered in numerical simulations ($L\leqslant 100$), this apparent asymptotic regime is still noticeably influenced by defect and network size effect corrections. This type of result is quite general and also applies in the mechanical context. The underlying explanation stems from the long range nature of the solutions of the Laplacian equation $\Delta V=0$. Hence, before identifying a singularity in a finite specimen, one should quantify carefully the finite-size corrections. This is important for the understanding of the statistical properties in breakdown problems of percolation models for instance and also in order to formulate reliable criteria for standard fracture testing (Chona *et al* 1983).

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References

Bowie O L 1973 *Solutions of crack problems by mapping technique* (Groningen: Noordhoff)
Bui H D 1977 *Mécanique de la rupture* (Paris: Masson)
Chona R, Irwin G R and Sanford R J983 *Influence of specimen size and shape on the singularity-dominated zone* (Special Technical Publication 791 1) (American Society for Testing and Materials)
Duxbury P M, Beale P D and Leath P L 1986 *Phys. Rev. Lett.* **57** 1052
—— 1987a *Phys. Rev. Lett.* **59** 155 (Erratum)
Duxbury P M, Leath P L and Beale P D 1987b *Phys. Rev.* B **36** 367
Erdogan F 1983 'Stress intensity factors' *Transactions ASME* **50** 992
Jackson J D 1975 *Classical Electrodynamics* 2nd edn (New York: Wiley) p 75-94
Kahng B, Batrouni G G, Redner S, de Arcangelis L and Herrmann H J 1988 *Phys. Rev.* B **37** 7625
Li Y S and Duxbury P M 1987 *Phys. Rev.* B **36** 5411
Sih G C 1973 *Mechanics of Fracture 1: Methods of Analysis and Solutions of Crack Problems* (Groningen: Noordhoff)